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ERRATUM TO "ORTHOGONAL CALCULUS"

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The proof of Theorem 6.3 in my paper *Orthogonal calculus* [W] contains a gap. This is caused by an error in the preliminaries [W, 6.2] ; the offending statement is ... *and happens to be inverse to* $\rho_{T(b)}$. The purpose of this note is to fill the gap.

Notation. $\mathcal{J}$ is the category of finite dimensional real vector spaces with a positive definite inner product. Morphisms in $\mathcal{J}$ are the linear maps respecting the inner product. $\mathcal{E}$ is the category of continuous functors from $\mathcal{J}$ to spaces. (The *spaces* in question are assumed to be compactly generated Hausdorff, homotopy equivalent to CW–spaces). A morphism $E \to F$ (natural transformation) in $\mathcal{E}$ is an *equivalence* if $E(V) \to F(V)$ is a homotopy equivalence for each V in $\mathcal{J}$. An object E in $\mathcal{E}$ is *polynomial of degree* $\leq n$ if, for each V in $\mathcal{J}$, the canonical map

$$\rho : E(V) \longrightarrow \operatorname*{holim}_{0 \neq U \subset \mathbb{R}^{n+1}} E(U \oplus V)$$

is a homotopy equivalence. The codomain of ρ, which we also denote by $(\tau_n E)(V)$, is a topological homotopy (inverse) limit [W, 5.1] ; more details below, in the proof of Lemma e.3. To repeat, E is polynomial of degree $\leq n$ if and only if $\rho : E \to \tau_n E$ is an equivalence.

6.3. Theorem. *For any $n \geq 0$, there exist a functor $T_n : \mathcal{E} \longrightarrow \mathcal{E}$ taking equivalences to equivalences, and a natural transformation $\eta_n : 1 \longrightarrow T_n$ with the following properties:*

1. *$T_n(E)$ is polynomial of degree $\leq n$, for all E in $\mathcal{E}$.*
2. *if E is already polynomial of degree $\leq n$, then $\eta_n : E \longrightarrow T_n E$ is an equivalence.*
3. *For every E in $\mathcal{E}$, the map $T_n(\eta_n) : T_n E \longrightarrow T_n T_n E$ is an equivalence.*

What we have to re–prove is 1. The remainder of the proof of 6.3 in [W] is not affected by the error in 6.2. As in [W] define $T_n E$ as the homotopy colimit (telescope in this case) of the direct system

$$E \xrightarrow{\ \rho\ } \tau_n E \xrightarrow{\tau_n(\rho)} \tau_n^2 E \xrightarrow{\tau_n^2(\rho)} \tau_n^3 E \xrightarrow{\tau_n^3(\rho)} \cdots . \tag{e.1}$$

It would be equally reasonable to define $T_n E$ as the homotopy colimit of

$$E \xrightarrow{\ \rho\ } \tau_n E \xrightarrow{\ \rho\ } \tau_n^2 E \xrightarrow{\ \rho\ } \tau_n^3 E \xrightarrow{\ \rho\ } \cdots \tag{e.2}$$

where the k–th map in the direct system is $\rho : \tau_n^{k-1} E \to \tau_n(\tau_n^{k-1} E)$. It turns out that the homotopy colimits of (e.1) and (e.2) are isomorphic, even relative to E. Namely, the Fubini principle for homotopy limits gives

$$(\tau_n^k E)(V) \cong \operatorname*{holim}_{0 \neq U_1, \ldots, U_k \subset \mathbb{R}^{n+1}} E(U_1 \oplus \cdots \oplus U_k \oplus V)\,.$$

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Using this as an identification and inspecting the maps in the direct systems (e.1) and (e.2), one finds that the direct systems are isomorphic.

e.3. Lemma. *Let $p : G \to F$ be a morphism in $\mathcal{E}$. Suppose that there exists an integer b such that $p : G(W) \to F(W)$ is $((n+1)\dim(W) - b)$–connected for all W in $\mathcal{J}$. Then $\tau_n(p) : \tau_n G(W) \to \tau_n F(W)$ is $((n+1)\dim(W) - b + 1)$–connected for all W.*

Proof. We begin with a discussion of the homotopy limits involved. Suppose first that Z is any functor from the poset $\mathcal{D}$ of nonzero linear subspaces of $\mathbb{R}^{n+1}$ to spaces. Ignoring the topology on $\mathcal{D}$, we can define holim Z as the totalization of the incomplete cosimplicial space

$$[k] \ \mapsto \prod_{L:[k]\hookrightarrow\mathcal{D}} Z(L(k)) \tag{e.4}$$

where L runs over the order–preserving *injections* from the poset $[k] = \{1, \ldots, k\}$ to $\mathcal{D}$. (An incomplete cosimplicial space is a covariant functor from the category with objects $[k]$ for $k \geq 0$ and monotone *injections* as morphisms to the category of spaces ; the totalization of such a thing is the space of natural transformations to it from the functor $[k] \mapsto \Delta^k$.)

We could make (e.4) into a complete cosimplicial space by dropping the injectivity condition on the order–preserving maps L ; the totalization would not change. However, totalizations of incomplete cosimplicial spaces are usually easier to understand than totalizations of complete cosimplicial spaces.— In (e.4) it is understood that a product $\prod_{i\in S}$ with empty S is a single point $*$; therefore the right–hand side of (e.4) is a point for $k > n + 1$.

Remembering the topology on $\mathcal{D}$ now, we note that $\mathcal{D}$ is a union of Grassmannians. Let us suppose that the spaces $Z(U)$ are the fibers of a fiber bundle ξ on $\mathcal{D}$ (that is, $Z(U)$ is the fiber over $U \in \mathcal{D}$), and that maps $Z(U_1) \to Z(U_2)$ induced by inclusions $U_1 \subset U_2$ depend continuously on U_1, U_2. Then it is appropriate to replace the incomplete cosimplicial space (e.4) by another incomplete cosimplicial space,

$$[k] \ \mapsto \Gamma(e_k^*\xi) \tag{e.5}$$

where e_k is the evaluation map $L \mapsto L(k)$, with domain equal to the space of monotone injections $L : [k] \to \mathcal{D}$, and codomain $\mathcal{D}$. The symbol Γ denotes a section space. The totalization of (e.5) is the *topological* homotopy limit of Z. For us, the relevant examples are $Z(U) := G(U \oplus W)$ and $Z(U) := F(U \oplus W)$ where W is fixed ; the topological homotopy limits are then $\tau_n G(W)$ and $\tau_n F(W)$, respectively.

The space of monotone injections $[k] \to \mathcal{D}$ is a disjoint union of manifolds $C(\lambda)$. Here $\lambda : [k] \to [n+1]$ is a monotone injection avoiding the element $0 \in [n+1]$, and $C(\lambda)$ consists of those $L : [k] \to \mathcal{D}$ for which $L(i)$ has dimension $\lambda(i)$. Writing $\lambda_i = \lambda(i)$ we find

$$\begin{aligned}
\dim(C(\lambda)) &= (n+1-\lambda_k)\lambda_k + \sum_{i=0}^{k-1}(\lambda_{i+1} - \lambda_i)\lambda_i \\
&= (n+1)\lambda_k + \sum_{i=0}^{k-1}\lambda_i\lambda_{i+1} \ - \sum_{i=0}^{k}\lambda_i^2 \\
&< (n+1)\lambda_k - k\,.
\end{aligned}$$

We see from (e.5) that the connectivity of $\tau_n(p) : \tau_n G(W) \to \tau_n F(W)$ is greater than or equal to the minimum of the numbers

$$(\text{connectivity of } p : G(L(k) \oplus W) \to F(L(k) \oplus W)) \ - \dim(C(\lambda)) - k$$

taken over all triples (L, λ, k) with $L \in C(\lambda)$ and $\lambda : [k] \to [n{+}1]$. By our hypothesis on $p : G \to F$, the connectivity of $p : G(L(k) \oplus W) \to F(L(k) \oplus W))$ is at least equal to $(n+1)(\lambda_k + \dim(W)) - b$. By the inequality for $\dim(C(\lambda))$, the minimum in question is greater than $(n+1)\dim(W) - b$. $\square$

Remark. The hypothesis in Lemma e.3 is strongly reminiscent of what Goodwillie in his calculus calls *agreement to n–th order*, in [Go3] and (for $n = 1$) in [Go1, 1.13]. Goodwillie also has lemmas similar to e.3, such as [Go1, 1.17] and [Go3, 1.6].

We fix some V in $\mathcal{J}$ from now on ; the goal is to prove that ρ from $T_nE(V)$ to $\tau_n(T_nE)(V)$ is a homotopy equivalence for any E in $\mathcal{E}$.

For W in $\mathcal{J}$ let $\mathrm{mor}(V, W)$ be the space of morphisms $V \to W$ in $\mathcal{J}$ and let $\gamma_1(V, W)$ be the Riemannian vector bundle on $\mathrm{mor}(V, W)$ whose total space is the set of (f, x) in $\mathrm{mor}(V, W) \times W$ with $x \perp \mathrm{im}(f)$. Let $\gamma_{n+1}(V, W)$ be the Whitney sum of $n+1$ copies of $\gamma_1(V, W)$, and let $S\gamma_{n+1}(V, W)$ be the unit sphere bundle of $\gamma_{n+1}(V, W)$. We abbreviate

$$F(W) := \mathrm{mor}(V, W),$$
$$G(W) := S\gamma_{n+1}(V, W)$$

and write $p : G \to F$ for the projection. By [W, 4.2, 5.2] the object G in $\mathcal{E}$ co–represents the functor $E \mapsto \tau_n E(V)$ from $\mathcal{E}$ to spaces. In more detail, writing $\mathrm{nat}(\dots)$ for spaces of natural transformations, we have a commutative diagram, natural in E:

$$\text{(e.6)} \qquad \begin{array}{ccc} E(V) & \xrightarrow{\ \rho\ } & \tau_n E(V) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{nat}(F, E) & \xrightarrow{\ p^*\ } & \mathrm{nat}(G, E)\,. \end{array}$$

e.7. Lemma. *$T_n p : T_n G \to T_n F$ is an equivalence.*

Proof. It is clear that $p : G \to F$ satisfies the hypothesis of Lemma e.3 with b equal to $(n+1)\dim(V) + 1$. (Here V is *not* a variable ; we fixed it, and used it in the definition of G and F.) Repeated application of Lemma e.3 shows that the connectivity of

$$\tau_n^k(p) : \tau_n^k G(W) \to \tau_n^k F(W)$$

tends to infinity as k goes to infinity, for any W in $\mathcal{J}$. Therefore $T_n p$ is an equivalence. $\square$

We shall use (e.7) to prove that the commutative square

$$\text{(e.8)} \qquad \begin{array}{ccc} E(V) & \xrightarrow{\ \subset\ } & T_n E(V) \\ \downarrow{\rho} & & \downarrow{\rho} \\ \tau_n E(V) & \xrightarrow{\ \subset\ } & \tau_n(T_n E)(V) \end{array}$$

can be enlarged to a commutative diagram of the form

$$\text{(e.9)} \qquad \begin{array}{ccccc} E(V) & \longrightarrow & X & \longrightarrow & T_nE(V) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \rho} \\ \tau_n E(V) & \longrightarrow & Y & \longrightarrow & \tau_n(T_nE)(V) \end{array}$$

in which the map g is a homotopy equivalence. (That is, (e.8) is obtained from (e.9) by deleting the middle column.) According to (e.6), diagram (e.8) is isomorphic to

$$\text{(e.10)} \qquad \begin{array}{ccc} \mathrm{nat}(F, E) & \xrightarrow{\subset} & \mathrm{nat}(F, T_nE) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ \mathrm{nat}(G, E) & \xrightarrow{\subset} & \mathrm{nat}(G, T_nE) \end{array}$$

and clearly (e.10) can be enlarged to

$$\text{(e.11)} \qquad \begin{array}{ccccc} \mathrm{nat}(F, E) & \longrightarrow & \mathrm{nat}(T_nF, T_nE) & \xrightarrow{\mathrm{res}} & \mathrm{nat}(F, T_nE) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle (T_np)^*} & & \downarrow{\scriptstyle p^*} \\ \mathrm{nat}(G, E) & \longrightarrow & \mathrm{nat}(T_nG, T_nE) & \xrightarrow{\mathrm{res}} & \mathrm{nat}(G, T_nE) \end{array}$$

where the arrows labelled res are restriction maps. We are now very close to having constructed a diagram like (e.9). The idea is that since $T_np : T_nG \to T_nF$ is an equivalence by Lemma e.7, the middle arrow in (e.11) ought to be a homotopy equivalence. Of course, it does not work exactly like that.

What is needed here is the notion of *cofibrant object in* $\mathcal{E}$ from the appendix of [W]. If $v : A \to B$ is an equivalence in $\mathcal{E}$ where A and B are cofibrant, then v admits a homotopy inverse $u : B \to A$, with (natural) homotopies relating vu and uv to the respective identity maps. Every object in $\mathcal{E}$ is the codomain of an equivalence whose domain is a so–called CW–object [W, A.4], and CW–objects are cofibrant [W, A.3]. More generally, every morphism $w : C \to D$ in $\mathcal{E}$ has a factorization

$$C \hookrightarrow D^\diamond \to D$$

where $D^\diamond \to D$ is an equivalence and $D^\diamond$ is a CW–object *relative to* D. (I leave definition and proof to the reader.) This factorization can be constructed functorially in $w : C \to D$, and if C is already cofibrant, then $D^\diamond$ will be cofibrant.

We apply this with w equal to the inclusion $F \to T_nF$ or to the inclusion $G \to T_nG$. It follows from (e.6) that F and G are cofibrant. Therefore $(T_nF)^\diamond$ and $(T_nG)^\diamond$ in the factorizations

$$F \hookrightarrow (T_nF)^\diamond \to T_nF\,, \qquad G \hookrightarrow (T_nG)^\diamond \to T_nG$$

are cofibrant. Replacing T_nF and T_nG by $(T_nF)^\diamond$ and $(T_nG)^\diamond$ in (e.11) we obtain a commutative diagram

$$\text{(e.12)} \qquad \begin{array}{ccccc} \mathrm{nat}(F, E) & \longrightarrow & \mathrm{nat}((T_nF)^\diamond, T_nE) & \xrightarrow{\mathrm{res}} & \mathrm{nat}(F, T_nE) \\ \downarrow{\scriptstyle p^*} & & \downarrow & & \downarrow{\scriptstyle p^*} \\ \mathrm{nat}(G, E) & \longrightarrow & \mathrm{nat}((T_nG)^\diamond, T_nE) & \xrightarrow{\mathrm{res}} & \mathrm{nat}(G, T_nE) \end{array}$$

and now the middle arrow is a homotopy equivalence. Diagram (e.12) is the explicit form or fulfillment of (e.9).

Proof of 1 in 6.3. We have to show that $\rho : T_nE(V) \to \tau_n(T_nE)(V)$ is a homotopy equivalence. It is enough to show that the vertical arrows in the commutative diagram

$$\text{(e.13)} \qquad \begin{array}{ccccccccc} E(V) & \xrightarrow{\rho} & \tau_nE(V) & \xrightarrow{\rho} & \tau_n^2E(V) & \xrightarrow{\rho} & \tau_n^3E(V) & \xrightarrow{\rho} & \cdots \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \\ \tau_nE(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^2E(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^3E(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^4E(V) & \xrightarrow{\tau_n(\rho)} & \cdots \end{array}$$

induce a map between the homotopy colimits of the rows which is a homotopy equivalence. It is enough because τ_n commutes with homotopy colimits over $\mathbb{N}$ up to homotopy equivalence, and because we can define T_nE as the homotopy colimit of (e.2). Denote the homotopy colimits of the rows in (e.13) by P and Q, and the map under investigation by $r : P \to Q$. For each $i \geq 0$ the commutative diagram

$$\begin{array}{ccc} \tau_n^iE(V) & \xrightarrow{\subset} & P \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle r} \\ \tau_n^{i+1}E(V) & \xrightarrow{\subset} & Q \end{array}$$

can be enlarged, as in (e.9) and (e.12), to a commutative diagram

$$\begin{array}{ccccc} \tau_n^iE(V) & \longrightarrow & X & \longrightarrow & P \\ \downarrow{\scriptstyle\rho} & & \downarrow & & \downarrow{\scriptstyle r} \\ \tau_n^{i+1}E(V) & \longrightarrow & Y & \longrightarrow & Q \end{array}$$

where the middle vertical arrow is a homotopy equivalence. It follows easily that $r : P \to Q$ is a homotopy equivalence. $\square$

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